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RECONSTRUCTION OF PLANE CRACKS IN AN ANISOTROPIC ELASTIC SOLID†

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Two methods are proposed for identifying plane cracks in an anistropic elastic medium, based either on the use of a "non-reciprocity" functional or on the non-classical method of boundary integral equations and leading to the solution of certain transcendental equations. Examples of the reconstruction of rectilinear cracks in a square region are examined.

In problems of identifying cracks in an elastic solid from the field on the surface of the solid, different versions of the boundary integral equation method are generally used as the basis for obtaining resolving operator equations. On the basis of this method it is possible to formulate systems of non-linear operator equations which are solved by iteration schemes [1–3]. In recent years, approaches to identifying cracks, based on the "non-reciprocity" functional (for the Laplace equation in recording thermal or electric fields) and *a priori* information about the crack (or system of cracks) situated in a certain plane, have begun to be developed intensively [4–7]. The availability of such *a priori* information considerably simplifies the procedure for reconstructing cracks and, for determining the parameters of a plane, leads either to a system of transcendental equations or to the problem of minimizing a certain non-quadratic discrepancy functional. Note that the calculation of wave fields in elastic solids based on the reciprocity theorem has undergone rapid development in recent years [8, 9].

Two methods of determining the position of a plane with cracks are examined below: the first involves formulating the "non-reciprocity" functional and selecting trial solutions, and the second is based on non-classical boundary integral equations for anisotropic solids with cracks, generalizing earlier results [10–12] for solids without defects.

1. FORMULATION OF THE PROBLEM

Suppose an elastic solid occupying a finite, simply connected region V with a boundary S performs steady oscillations with an angular velocity ω, the body being weakened by a system of cracks

$$\Gamma = \bigcup_{p=1}^{M} \Gamma_p \; (\Gamma_p = \Gamma_p^+ \cup \Gamma_p^-)$$

which are modelled by cuts in the cross-section S_c of the region V with a certain surface Π.

The equations of steady oscillations have the form [13]

$$\sigma_{ij,j} = -\rho\omega^2 u_i, \quad \sigma_{ij} = c_{ijkl}u_{k,l}, \quad i = 1, 2, 3, \quad \underline{x} \in V\backslash\Gamma \tag{1.1}$$

We will assume that, in the direct problem, the stress vector on the external boundary S is specified, and the sides of the cracks do not interact with each other during the oscillations

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$$t_i|_S = \sigma_{ij}n_j|_S = \phi_i, \quad t_i|_{\Gamma_p^{\pm}} = 0, \quad p = 1, 2, \ldots, M \tag{1.2}$$

In formulating the inverse problem, in which the equation of the surface Π and the crack geometry are determined, we will assume that, on the boundary S, all components of the displacement vector are known

$$u_i|_S = \psi_i, \tag{1.3}$$

Remark. In principle, to solve the inverse problem it is sufficient to know the displacement field on part of the boundary, and, to extend the field to the entire boundary of the solid, the procedure proposed earlier [14–16] is used.

2. RECORDING OF CRACKS

Cracks in a solid can be found from the discrepancy between the boundary wave fields of stresses and displacements for the solid with and without defects. As a measure of this discrepancy, we will introduce into consideration the linear functional

$$F(\underline{u}^*, \omega) = G(\phi, \psi, \underline{u}^*) = \int_S (\psi_i\phi_i^* - \psi_i^*\phi_i)dS \tag{2.1}$$

on a set of trial solutions u_i^* satisfying the equations of motion in the region V without cracks

$$\sigma_{ij,j}^* = -\rho\omega^2 u_i^*, \quad \sigma_{ij}^* = c_{ijkl}u_{k,l}^* \quad i = 1, 2, 3, \quad \underline{x} \in V \tag{2.2}$$

where ψ_i^* and ϕ_i^* are components of the displacement vector and stress vector of the trial solution on the boundary S

$$u_1^*|_S = \psi_i^*, \quad t_i^*|_S = \sigma_{ij}^* n_j|_S = \phi_i^* \tag{2.3}$$

Note that plane waves in an unbound elastic medium can be used as solutions of this kind.

Using the reciprocity theorem [13], it can be shown that

$$\int_{\Gamma^+} \chi_i q_i^* dS = G(\underline{\phi}, \underline{\psi}, \underline{u}^*) = F(\underline{u}^*, \omega) \tag{2.4}$$

where $\chi_i = u_i|_{\Gamma^+} - u_i|_{\Gamma^-}$ are the displacement jumps on cracks, and $q_i^*|_{\Gamma^+} = \sigma_{ij}^* n_j|_{\Gamma^+}$ are components of the vector of trial stresses on the sides of cracks.

Where there are no cracks, the functional $F(\underline{u}^*, \omega)$ is identically equal to zero, since relation (2.4) is transformed into the well-known reciprocity relation. Note that the functional (2.1) in the present formulation, where the strain and stress fields are known on the entire boundary of the solid, can be calculated for any trial field. Thus, knowing the accuracy of the measurement of the boundary fields and the accuracy of the calculation of the integral (2.1), from the deviation of the functional from a zero value it is possible to judge the presence or absence of cracks in the body. The inverse problem is solved most simply when *a priori* information indicates that the surface Π is a plane.

3. SELECTION OF TRIAL SOLUTIONS AND DETERMINATION OF THE PLANE Π

An anisotropic elastic solid. From Eq. (2.4), by an appropriate selection of trial solutions possessing a sufficient functional arbitrariness, a system of integral equations can be obtained for finding displacement jumps on cracks. The first stage of this procedure is to determine the plane Π, which can be specified by three independent parameters (for example, two Euler angles and the distance c to the origin of coordinates). The trial solutions must be selected in such a way that the parameters characterizing the plane Π are found most simply, and here one of the methods of making such a selection leads to the elimination of unknown displacement jumps.

For this, we will introduce a new system of coordinates $O\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3$, in which the equation of the plane Π has the form $\mathbf{X}_3 = c$. The position of such a system of coordinates with respect to the initial system is defined by the two angles φ and θ: the angle φ gives the rotation about the axis Ox_3, and the angle θ gives the rotation about the axis $O\mathbf{X}_1$.

The notation U_i and U_i^* $(i = 1, 2, 3)$ will be used for the vector components of the solution of boundary-values problem (1.1), (1.2) and the trial solution (2.2), (2.3) in the new system of coordinates. The left-hand side of relation (2.4) in the adopted notation will take the form

$$\int\limits_{\Gamma^+} \chi_i q_i^* dS = \int\limits_{\Gamma^+} \mathrm{X}_i Q_i^* dS \tag{3.1}$$

where

$$\mathrm{X}_i = U_i\big|_{\Gamma^+} - U_i\big|_{\Gamma^-}, \quad Q_i^* = \sigma^*_{\mathbf{X}_i\mathbf{X}_3} \tag{3.2}$$

As trial solutions we will select a system of functions comprising waves propagating along $\mathbf{X}_3$ axis.

$$U_k^* = A_k \exp(i\alpha \mathbf{X}_3) \tag{3.3}$$

Substituting expressions (3.3) into Eqs (2.2), from the condition for non-zero trial solutions for the wave parameter α to exist we will obtain the well-known Christoffel equation [13], which has three pairs of solutions – real manifolds (among which multiple values may also be encountered)

$$\alpha_j^{\pm} = \pm\alpha_j(c_{iklm}, \varphi, \theta), \quad j = 1, 2, 3 \tag{3.4}$$

corresponding to different types of wave propagating in an infinite elastic anisotropic medium. Substitution of the corresponding set of six solutions (3.3), (3.4) into relation (2.4) leads to a system of six algebraic equations in φ, θ, c and I_1, I_2, I_3 $\left(I_i = \int_{\Gamma^+} X_i dS\right)$ in which the quantities I_i occur linearly, which enables us to eliminate them and to obtain three transcendental equations in the parameters of the plane Π.

An isotropic elastic solid. For an isotropic medium this combination of six equations has a fairly simple form, corresponding to a longitudinal wave and two transverse waves

$$\begin{aligned}
&\underline{U}^{*1} = (\cos k_2\mathbf{X}_3, 0, 0), \quad \underline{U}^{*2} = (\sin k_2\mathbf{X}_3, 0, 0)\\
&\underline{U}^{*3} = (0, \cos k_2\mathbf{X}_3, 0), \quad \underline{U}^{*4} = (0, \sin k_2\mathbf{X}_3, 0)\\
&\underline{U}^{*5} = (0, 0, \cos k_1\mathbf{X}_3), \quad \underline{U}^{*6} = (0, 0, \sin k_1\mathbf{X}_3)
\end{aligned} \tag{3.5}$$

where $k_1 = \sqrt{\rho\omega^2/(\lambda + 2\mu)}$ and $k_2 = \sqrt{\rho\omega^2/\mu}$ are the longitudinal and transverse wave numbers, respectively. The vectors of the trial stresses in the cross-section S_c correspondingly have the forms

$$\begin{aligned}
&\underline{Q}^{*1} = (-\mu k_2 \sin k_2 c, 0, 0), \quad \underline{Q}^{*2} = (\mu k_2 \cos k_2 c, 0, 0)\\
&\underline{Q}^{*3} = (0, -\mu k_2 \sin k_2 c, 0), \quad \underline{Q}^{*4} = (0, \mu k_2 \cos k_2 c, 0)\\
&\underline{Q}^{*5} = (0, 0, -(\lambda + 2\mu)k_1 \sin k_1 c), \quad \underline{Q}^{*6} = (0, 0, -(\lambda + 2\mu)k_1 \cos k_1 c)
\end{aligned} \tag{3.6}$$

Substituting expressions (3.6) into relations (2.4), we obtain six equations in the six unknown quantities φ, θ, c, I_1, I_2 and I_3

$$\begin{aligned}
&-I_1\mu k_2 \sin k_2 c = G(\phi, \underline{\psi}, \underline{U}^{*1}), \quad I_1\mu k_2 \cos k_2 c = G(\phi, \underline{\psi}, \underline{U}^{*2})\\
&-I_2\mu k_2 \sin k_2 c = G(\phi, \underline{\psi}, \underline{U}^{*3}), \quad I_2\mu k_2 \cos k_2 c = G(\phi, \underline{\psi}, \underline{U}^{*4})
\end{aligned} \tag{3.7}$$

$$-I_3(\lambda+2\mu)k_1\sin k_1 c = G(\underline{\phi}, \underline{\psi}, \underline{U}^{*5}), \quad I_3(\lambda+2\mu)k_1\cos k_1 c = G(\underline{\phi}, \underline{\psi}, \underline{U}^{*6})$$

Assuming that I_1, I_2 and I_3 are not equal to zero, we reduce system (3.7) to three equations in φ, θ and c that define plane Π

$$F_{12}(\varphi, \theta) - F_{34}(\varphi, \theta) = 0 \tag{3.8}$$

$$\mathrm{tg}k_2 c = F_{34}(\varphi, \theta), \quad \mathrm{tg}k_1 c = F_{56}(\varphi, \theta) \tag{3.9}$$

where

$$F_{ij}(\varphi, \theta) = -G(\underline{\phi}, \underline{\psi}, \underline{U}^{*i})/G(\underline{\phi}, \underline{\psi}, \underline{U}^{*j})$$

Plane strain of an isotropic elastic solid. In the case of plane strain (in the plane $O\mathbf{X}_2\mathbf{X}_3$), the position of the plane Π is defined by the angle θ and the distance c. These parameters, together with integral jumps I_2 and I_3, are connected by four equations of the form (3.7) which, after eliminating the parameters I_2 and I_3, reduce to a system of equations of the form (3.9) with $\varphi = 0$.

If d is the diameter of the cross-section (in the plane $O\mathbf{X}_2\mathbf{X}_3$) of the region V, then, for the oscillation frequencies

$$f = \frac{\omega}{2\pi} < \frac{1}{4d}\sqrt{\frac{\mu}{\rho}}$$

this system reduces to the single equation

$$\mathrm{arctg}F_{34}(0, \theta) - \sqrt{\frac{2(1-\nu)}{1-2\nu}}\,\mathrm{arctg}F_{56}(0, \theta) = 0, \quad \theta \in [0, \pi] \tag{3.10}$$

after solving which the distance c is found from one of the formulae

$$c = k_1^{-1}\mathrm{arctg}F_{56}(0, \theta) \quad \text{or} \quad c = k_2^{-1}\mathrm{arctg}F_{34}(0, \theta) \tag{3.11}$$

4. DETERMINATION OF THE PLANE c USING A BOUNDARY INTEGRAL EQUATION OF THE FIRST KIND

Reduction of boundary-value problem (1.1), (1.2) to a boundary integral equation of the first kind. For finite isotropic solids, a method has been proposed [10] for formulating a system of boundary integral equations of the first kind with smooth kernels for cracked solids that does not require a knowledge of the fundamental solutions of the theory of elasticity operator. This approach has been extended [11, 12] to finite anisotropic solids without defects. We will generalize this result for the case of an anisotropic solid with a crack or a system of cracks located on a certain surface.

Using the approach of the theory of dislocations [13], problem (1.1), (1.2) is equivalent to the boundary-value problem for a continuous solid V with mass forces

$$f_i = -(c_{ijkl}n_k\chi_l\delta(\zeta))_{,j}$$

concentrated on the surface Γ, where n_k represents the vector components of the normal to the surface Γ, $\delta(\zeta)$ is the Dirac delta function and ζ is the coordinate along the normal to Γ.

We multiply the equations of motion by $e^{i(\alpha, x)}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and integrate over V. Using Gauss' theorem, to find the Fourier transform

$$\tilde{u}_k(\alpha) = \int\limits_V u_k(x)e^{i(\alpha, x)}dV_x$$

we have the following system of equations

$$A_{sk}(\alpha)\tilde{u}_k = (c_{sjkl}\alpha_j\alpha_l - \rho\omega^2\delta_{sk})\tilde{u}_k(\alpha) = F_s(\alpha) + W_s(\alpha) \tag{4.1}$$

where

$$\begin{aligned} F_s(\alpha) &= \int_S (\sigma_{sj}n_j - i\alpha_j c_{sjkl}u_k n_l)e^{i(\alpha, x)}dS_x \\ W_s(\alpha) &= -i\alpha_j c_{sjkl}\int_\Gamma n_k^0\chi_l e^{i(\alpha, x)}dS_x \end{aligned} \tag{4.2}$$

Solving system (4.1) for the transform $\tilde{u}_k$, we obtain

$$\tilde{u}_k(\alpha) = \frac{p_{ks}(\alpha)(F_s(\alpha) + W_s(\alpha))}{p_0(\alpha)}, \quad p_0(\alpha) = \det A \tag{4.3}$$

where $p_{ks}(\alpha)$ are components of the matrix associated to the matrix $A = \{A_{ks}\}$. Note that $p_0(\alpha)$ is a sixth-degree polynomial, $p_{ks}(\alpha)$ is a fourth-degree polynomial in α and $p_0(\alpha) = 0$ is Christoffel's equation, which has three roots

$$\gamma = |\alpha| = k\varsigma_m(\eta), \quad m = 1, 2, 3, \quad (|\eta| = 1, k = \omega(c_{3333}/\rho)^{-1/2})$$

corresponding to three surfaces which below will be assumed to be different (although this limitation is not always satisfied; for example, it is not satisfied in the isotropic case when two of the surfaces coincide).

The Fourier transforms $\tilde{u}_k(\alpha)$, by virtue of representation (4.3), are mesomorphic functions, but, on the other hand, it is well known that the Fourier transform of the function from $L_p(V)$, $p > 1$ is an integral function. Thus, it is necessary for the numerator on the right-hand side of Eq. (4.3) to vanish on the sets

$$\alpha^{(m)}(\eta) = k\varsigma_m(\eta)\eta, \quad m = 1, 2, 3$$

which leads to a system of resolvents, three of which are independent, for example the following

$$p_{1s}(\alpha^{(m)}(\eta))(F_s(\alpha^{(m)}(\eta)) + W_s(\alpha^{(m)}(\eta))) = 0; \quad |\eta| = 1, \quad m = 1, 2, 3 \tag{4.4}$$

Equalities (4.4) can be treated as a system of boundary equations relating boundary displacements and stresses to displacement jumps on a crack. Note that this system of equations is suitable for carrying out the procedure of reconstructing plane cracks.

Determination of the plane Π. Assuming all the vector components of the displacements and stress vectors on the entire external boundary of the solid are known, the operators $F_s(k\varsigma_m(\eta)\eta)$ can be calculated for any unit vector η. Thus, from Eqs (4.4) it is necessary to determine the jumps χ_l and the carrier of the crack – the surface Γ. When *a priori* information is available indicating that Γ is a plane region ($\Gamma \subset \Pi$), it is possible to compare the equations for determining the parameters characterizing the plane Π without finding the jumps χ_l.

Let $x = c\eta_0 + \eta_1 t_1 + \eta_2 t_2$ be the parametric equation of the plane Π, where

$$|\eta_0| = |\eta_1| = |\eta_2| = 1, \quad (t_1, t_2) \in D, \quad (\eta_0, \eta_1) = (\eta_0, \eta_2) = 0, \quad n^0 = \eta_0$$

We will introduce into consideration the functions

$$\begin{aligned} G_m(\eta, t_1, t_2) &= (\alpha^{(m)}(\eta), x) = k\varsigma_m(\eta)(\eta, c\eta_0 + \eta_1 t_1 + \eta_2 t_2) = \\ &= k\varsigma_m(\eta)[c(\eta, \eta_0) + t_1(\eta, \eta_1) + t_2(\eta, \eta_2)] \end{aligned} \tag{4.5}$$

Assuming here that $\eta = \eta_0$, we find that

$$G_m(\eta_0, t_1, t_2) = k\varsigma_m^0 c, \quad \varsigma_m^0 = \varsigma_m(\eta_0)$$

and does not depend on t_1 and t_2. Thus, putting

$$\int_D \chi_l(x(t))dt_1 dt_2 = Y_l$$

from relations (4.4) we obtain that

$$W_s(k\varsigma_m^0\eta_0) = -ik\varsigma_m^0 c_{sjkl}\eta_{0j}\eta_{0k}\exp[i\vartheta_m^0]Y_l, \quad \vartheta_m^0 = k\varsigma_m^0 c$$

To find Y_l we obtain the linear algebraic system

$$\begin{aligned} &\sum_{l=1}^{3} D_{lm}Y_l = b_m \\ &D_{lm} = ikp_{1s}(k\varsigma_m^0\eta_0)\varsigma_m^0 c_{sjkl}\eta_{0j}\eta_{0k}, \quad b_m = \exp[-i\vartheta_m^0]p_{1s}(k\varsigma_m^0\eta_0)F_s(k\varsigma_m^0\eta_0) \end{aligned} \tag{4.6}$$

Taking into account that Y_l are real while D_{lm} are pure imaginary quantities, we obtain a system of three transcendental equations for determining the components of the unit vector η_0 and the constant c

$$\mathrm{Re}(b_m) = 0, \quad m = 1, 2, 3 \tag{4.7}$$

Plane deformation of an orthotropic solid. We will give the form of the equations for determining η_0 in the case of the plane deformation of an orthotropic solid with boundary L (in the $O\mathbf{X}_1\mathbf{X}_2$ plane). The corresponding operators and the curves $\varsigma_m^0 = \varsigma_m(\eta_0)$ ($m = 1, 2$) are given in [12]. We will introduce the angle φ_0 such that $\eta_0 = (\cos\varphi_0, \sin\varphi_0)$. Then, introducing the notation $\xi_m^0(x) = (\eta_0, x)k\varsigma_m^0$ and taking into account the property $p_{ij}(ky) = k^2 p_{ij}(y)$, we find

$$F_j(k\varsigma_m^0\eta_0) = \int_L \left\lfloor t_j(x) - ik\varsigma_m^0 a_{jk}^0 u_k(x)\right\rfloor \exp[i\xi_m^0(x)]dl_x, \quad j = 1, 3; \quad m = 1, 2$$

where

$$t_j(x) = \sigma_{jk}n_k/c_{33}, \quad a_{jk}^0 = a_{jk}(\eta_0, n)$$

$$a_{11}^0 = \gamma_1 n_1\sin\varphi_0 + \gamma_5 n_3\cos\varphi_0, \quad a_{13}^0 = \gamma_5 n_1\sin\varphi_0 + \gamma_7 n_3\cos\varphi_0$$

$$a_{31}^0 = \gamma_7 n_1\sin\varphi_0 + \gamma_5 n_3\cos\varphi_0, \quad a_{33}^0 = \gamma_5 n_1\cos\varphi_0 + n_3\sin\varphi_0$$

$$p_{11}(\varsigma_m^0\eta_0) = (\gamma_5\cos^2\varphi_0 + \sin^2\varphi_0)(\varsigma_m^0)^2 - 1$$

$$p_{13}(\varsigma_m^0\eta_0) = -(\gamma_5 + \gamma_7)\sin\varphi_0\cos\varphi_0(\varsigma_m^0)^2$$

(γ_j are dimensionless elasticity constants), while the system of two equations of the form (4.7) for finding φ_0 and c acquires the form

$$p_{1s}(\varsigma_m^0\eta_0)\int_L [t_s(x)\cos(\xi_m^0(x) - \vartheta_m^0) + k\varsigma_m^0 a_{sk}^0 u_k(x)\sin(\xi_m^0(x) - \vartheta_m^0)]dl_x = 0, \quad m = 1, 2 \tag{4.8}$$

5. NUMERICAL EXPERIMENTS

As an example of the application of the method for determining the position of a crack, that was proposed in the concluding part of Section 3 (Example 1), and an example of the use of system (4.8) (Example 2), we considered the steady oscillations (with a frequency $f = 1.0$ kHz) of a square with sides of 0.1 m under conditions of plane strain, weakened by a rectilinear crack. To solve the direct problem, we used the ACELAN finite-element package [17]; in the calculations, the finite element grid was artificially concentrated in the vicinity of the crack tips; the total number of nodes was 1181 in Example 1 and 1550 in Example 2. On the each side of the square, besides the nodes at its tips, a total of nine internal nodes was selected, equidistant from each other. The displacements found at these nodes modelled the process of measuring the boundary fields. When calculating functional (2.4), the factors of the integrands were interpolated by continuous piecewise-linear functions.

Example 1. *Identification of a crack in an isotropic square.* The coordinates of the ends of the crack were $A(0.02; 0.06)$ and $B(0.04; 0.08)$ (Fig. 1a). The material of the square was steel. On the bottom and top sides of the square, a balanced load was specified in the form of a normal stress (evenly distributed with intensity $Q_0 = 10^3$ N/m and varying linearly from zero to Q_0), and the left- and right-hand surfaces were load-free.

Figures 1(b) and (c), for the undeformed state of the region, show the distributions (with an isoline grid) of the amplitude values of the vector components of the displacements: the components u_2 (Fig. 1b), the maximum value of the amplitude, equal to 1.15×10^{-9} m and indicated with an asterisk, corresponding to the light background, and the minimum value, equal to -1.06×10^{-9} m and shown by a light point, corresponding to the dark background, and the components u_3 (Fig. 1c), the maximum value being equal to 2.98×10^{-9} m, and the minimum value to -2.45×10^{-9} m.

Note that, as a result of a numerical solution of non-linear equation (3.10), several of its roots were found, and, to select a single solution, it was necessary either to analyse the initial system in a certain set of frequencies or to change the nature of the load (the uniform stress distribution was changed to a linear distribution). In a numerical simulation, we investigated the influence of the size of the defect on the relative errors in determining the angle θ and the distance c

$$\delta_\theta = [(\theta_0 - \theta_n)/\theta_0] \times 100\%, \quad \delta_c = [(c_0 - c_n)/c_0] \times 100\%$$

where $\theta_0 = \pi/4$ and $c_0 = 0.04\sqrt{2}$ m are precise values, while θ_n and c_n are the values of the parameters obtained.

In the left-hand and lower parts of Fig. 2, the values of δ_c when the crack length L changes from 0.0035 to 0.2083 m are shown by the light points, and here the values of δ_θ (dark points) do not exceed 1% in modulus.

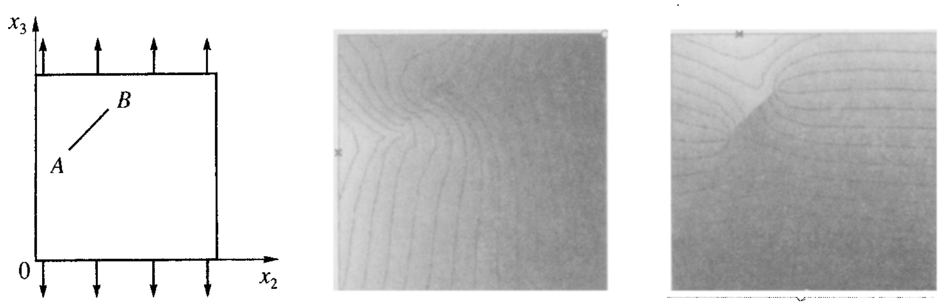
The check of the stability of the algorithm for crack reconstruction to random errors of the input data was modelled by pseudorandom perturbations of the amplitudes of "measured" quantities in such a way that

$$\tilde{u}_i(x_{(k)}) = u_i(x_{(k)})(1 + (2R_k - 1)\varepsilon)$$

where the perturbed values are indicated by a tilde, k is the number of the point on the boundary at which the "measurements' are carried out, R_k is a random quantity, evenly distributed in the interval (0, 1) and $\varepsilon = 10^m$ is a small parameter. The right-hand upper part of Fig. 2 shows the errors δ_c and δ_θ when the exponent m changes from –5 to –1; it can be seen that the order of the reconstruction error does not exceed the order of the error of the input data.

Example 2. *Identification of a crack in an orthotropic square.* The coordinates of the crack were A (0.02; 0.08) and B (0.04; 0.06) (Fig. 3a). The material of the square was austenitic steel [12]. A balanced load was specified on all sides of the square in the form of a uniformly distributed normal stress with intensity $Q_0 = 10^4$ N/m.

Figures 3(b) and (c), for the undeformed state of the region, show the distributions (with an isoline grid) of the amplitude values of the vector components of the displacements; the components u_1 (Fig. 3b), the maximum value of the amplitude, equal to 1.68×10^{-9} m and indicated with an asterisk, corresponding to the light background, and the minimum value, equal to -1.72×10^{-9} m and indicted with a light point, corresponding to the dark background, and the components u_3 (Fig. 3c), the maximum value being equal to 2.38×10^{-9} m and the minimum value to -2.29×10^{-9} m.



(a) (b) (c)

Fig. 1

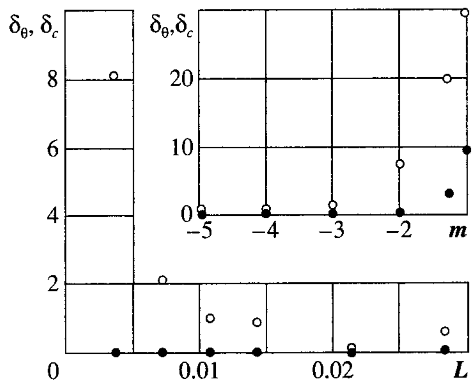
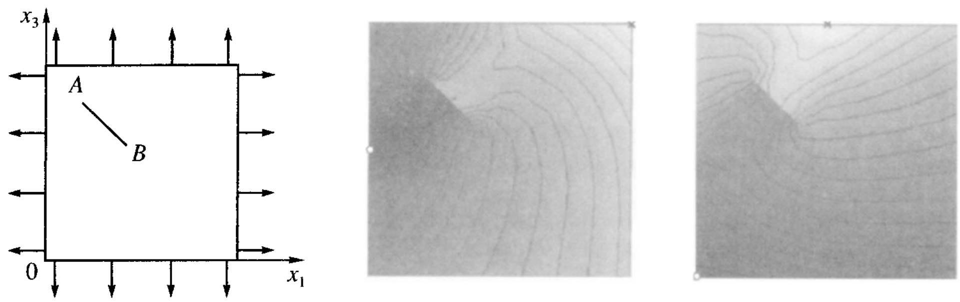


Fig. 2



(a) (b) (c)

Fig. 3

As a result of a numerical solution of the non-linear system of equations (4.8), several of its solutions were found, and here, for a unique determination of the true solution and to filter out "phantom" solutions, it was necessary either to change the nature of the load (to free the left- and right-hand sides from stresses) or to conduct a numerical simulation in a certain set of frequencies. The error of the values of the angle and distance obtained for $\varphi_0 = \pi/4$ and $c = 0.05\sqrt{2}$ m did not exceed 1%.

A check of the stability of the algorithm for crack reconstruction to errors of the input data led to results similar to the data for Example 1.

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